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From Worked Examples to Mathematical Principles: Modern Scientific Explanations of Suanfa Tongzong and Mental Abacus

Shulang Cao

Beanstalk International Bilingual School, Chaoyang District, Beijing, China

Abstract

Taking a square-root problem involving square bricks in Volume Six of Cheng Dawei's Ming-dynasty Xinbian Zhizhi Suanfa Tongzong as its starting point, this paper raises a question about the denominator "seventy-seven" in the answer. Through verification, a generalized derivation, and a function graph, it interprets the traditional rule for handling the remainder as linear interpolation of the square-root function and provides an exact error formula. It then examines the mathematical meaning of other algorithms in the ancient book, explaining the mathematical principles of abacus calculation through decimal place value, complement transformations, and updates to the computational state. Drawing on psychological experiments and neuroscience research, it discusses spatial representations and procedural operations in mental abacus calculation. The paper aims to show that traditional abacus calculation and mental abacus calculation can be explained through modern mathematics, computing science, and cognitive science, respectively. Finally, it discusses questions that remain to be investigated and reflects on the connections between traditional knowledge and modern science, connections that are both specific and bounded.

Keywords

Cheng Dawei, Suanfa Tongzong, Abacus Calculation, Mental Abacus, Linear Interpolation, Numerical Computation.

1. Starting with a Denominator: A Mathematical Investigation of the Square-Brick Square-Root Problem

Xinbian Zhizhi Suanfa Tongzong is a seventeen-volume practical mathematical treatise compiled by Cheng Dawei during the Ming dynasty. It covers abacus mnemonics, commercial calculations, land measurement, and root extraction, among other topics [1-2]. While reading the "Shaoguang Chapter" in Volume Six, I encountered a problem: "There are now one thousand four hundred and sixty-one square bricks. If they are to be arranged into a square, how many bricks will there be along one side?" The book answers: "Thirty-eight bricks along one side, and seventeen seventy-sevenths of a brick" [1]. Taking one brick as a unit of area, the problem is to find the side length of a square with an area of 1461. In modern notation, the answer in the ancient book is

$$A = 38 + \frac{17}{77}. \tag{1}$$

I first checked the answer by squaring it and found that $A^2 = 1461 - 1020/5929$, so it is not the exact square root. Why is the integer part 38, and why should the remainder 17 be divided by 77?

Determining the integer part is not difficult. Since $30^2 < 1461 < 40^2$, one can first try 3 in the tens place and subtract $30^2 = 900$, leaving 561. If the units digit is b, the additional area required is

$$(30 + b)^2 - 30^2 = 60b + b^2. \tag{2}$$

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Taking $b = 8$ adds 544; taking 9 instead would add 621, which exceeds the remaining area. The integer part is therefore 38, with a remainder of $1461 - 38^2 = 17$. Geometrically, the additional area consists precisely of two rectangles and a small square at a corner. This corresponds to the "fangfa," "lianfa," and "yufa" used in the ancient book's digit-by-digit square-root extraction: starting from the portion of the root already determined, the remaining area is subtracted through a combination of doubling that root and using the next trial digit [1].

Now consider the denominator: $39^2 - 38^2 = 77$. This suggested to me that the traditional method might use the position of 1461 between two consecutive perfect squares to estimate the position of its square root between 38 and 39. On consulting the explanation of "non-exact square-root extraction" in Volume Six, I found that the book does indeed give a rule that doubles the root already obtained and adds one to form the denominator [1]. This makes a generalized investigation possible.

Let a nonnegative integer be $N = q^2 + r$, where $q = \lfloor \sqrt{N} \rfloor$ and $0 \le r < 2q + 1$. This approximation algorithm can then be written as

$$A(N) = q + \frac{r}{2q+1}. \tag{3}$$

Why is Equation (3) reasonable? Let $f(x) = \sqrt{x}$, and consider the two adjacent points (q^2, q) and $((q+1)^2, q+1)$ on its graph. The slope of the line joining them is $1/(2q+1)$, so the value of this linear function at $x = N$ is exactly Equation (3). In modern numerical analysis, this is called linear interpolation: estimating the value of a function on a curve using the straight line between two known points [3]. The correspondence here concerns mathematical structure; it is not a claim that Cheng Dawei had already proposed modern interpolation theory.

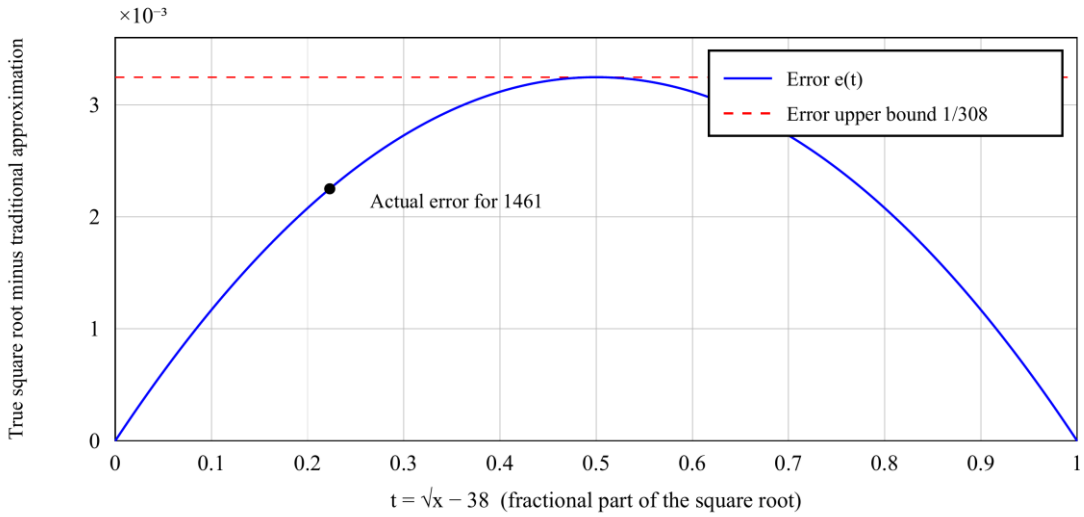


Figure 1: Error of square-root linear interpolation and its theoretical upper bound (q = 38).

Figure 1 plots the interpolation error separately, making it easier to observe how it varies with the fractional part of the square root. To draw a continuous curve, the N in the interpolation formula is temporarily extended here to a real number x in the interval $[38^2, 39^2]$; the original integer problem consists only of discrete values within it. However, the graph provides only an intuitive impression; the direction and magnitude of the error still require algebraic proof. Write $\sqrt{N} = q + t$, where $0 \le t < 1$; then $r = 2qt + t^2$. Substituting into Equation (3) and simplifying gives

$$\sqrt{N} - A(N) = \frac{t(1-t)}{2q+1}. \tag{4}$$

From $0 \le t(1-t) \le 1/4$, we immediately obtain

$$0 \le \sqrt{N} - A(N) \le \frac{1}{4(2q+1)}. \tag{5}$$

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This proves two things: first, the value obtained by the traditional method never exceeds the true square root; second, the error has an explicit upper bound. For $N = 1461$, the error is no greater than $1/308 \approx 0.003247$, while the actual error is approximately 0.00225049. The error curve in Figure 1 is a downward-opening parabola, which also explains why the error is zero at both ends of the interval and larger near the middle.

From finding 38, to understanding 17 and 77, and then to proving the interpolation relationship and the error bound, the initial question has now been fully answered. An ancient rule that appeared to have been devised merely to make calculation convenient turns out to admit quantitative testing through modern mathematics. This prompted me to ask a further question: what other methods in Suanfa Tongzong allow us to move from "how to calculate" to "why it works"?

2. From a Single Problem to an Entire Book: Mathematical Traces in Ancient Algorithms

As I continued reading Suanfa Tongzong, I found that its value lies not only in preserving many interesting mathematical problems, but also in handling different problems through steps that can be executed repeatedly. Volume One introduces basic methods such as place-value positioning and multiplication and division mnemonics; Volume Six discusses root extraction; and Volume Ten uses "yingnu," or excess and deficit, to determine unknown quantities from surpluses and shortfalls [1]. "Ying" means having too much, while "nu" means not having enough.

Volume Ten contains a problem about a group purchasing an item together: if each person contributes five taels of silver, there is a surplus of six taels; if each contributes three taels, there is a shortfall of four taels. The original book gives the answer as five people and a price of nineteen taels [1]. Let the number of people be x and the price be y; this yields $5x - y = 6$ and $3x - y = -4$. Subtracting the two equations gives $2x = 10$, hence $x = 5$ and $y = 19$. Although the original book does not use the notation of simultaneous linear equations in two unknowns, it can determine the unknown quantities by comparing the two surpluses and shortfalls. Today, we can understand this as the idea of elimination; similarly, digit-by-digit square-root extraction can be explained through the perfect-square identity. This does not mean that ancient practitioners and modern mathematicians used the same conceptual system, but rather that two forms of expression can refer to the same verifiable quantitative relationships.

These problems also drew my attention to another role of mnemonics: they not only aid memory but also prescribe the order of operations. An algorithm must tell the calculator what to do next, when to perform an operation, and when to stop. To understand this procedural way of calculating, we must study more than the answers on paper; we must also examine the tool to which Suanfa Tongzong attaches particular importance—the abacus.

3. Why the Abacus Can Calculate: Number Representation, Making Five, and Carrying Ten

The ingenuity of the abacus begins with the representation of numbers. Different rods correspond to the units, tens, and hundreds places; on a single rod, an upper bead next to the beam usually represents 5, while a lower bead represents 1. Historical abacuses did not all have the same design. For ease of analysis, the discussion below uses a normalized representation with at most one upper bead and four lower beads per rod, without treating this as the sole design of all Ming-dynasty abacuses [2].

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Let the counts of upper and lower beads next to the beam on rod i, counted from right to left, be u_i, l_i, respectively, where $u_i \in \{0,1\}$ and $l_i \in \{0,1,2,3,4\}$. The digit on that rod is then $d_i = 5u_i + l_i$. The nonnegative integer represented on the abacus is

$$N = \sum_{i=0}^{m-1}(5u_i + l_i)\,10^i, \quad 0 \le d_i \le 9. \tag{6}$$

Equation (6) contains two simple but crucial facts: the value of a digit depends on the combination of bead position and place value, and a multidigit number can be decomposed into local information on individual rods. For example, 85 is represented by 8 in the tens place and 5 in the units place. The beads themselves are not inherently 85; position and state jointly encode the number.

Representing numbers, however, is not yet calculation. I tried to perform $48 + 37$ according to abacus rules: first set 48 on the abacus, and split 37 into 30 and 7. When adding 30, the tens digit is initially 4 and must be increased by 3. Using $3 = 5 - 2$, one can add one upper bead and remove two lower beads, changing the tens digit to 7 and obtaining 78. When next adding 7, the units digit is initially 8 and a carry is required. Using $7 = 10 - 3$, one can carry 1 to the tens place while subtracting 3 in the units place, obtaining 85. These procedures illustrate making five and complementing to ten, respectively.

Table 1: Changes in abacus state when calculating 48 + 37.

Step	Tens Digit	Units Digit	Abacus Value	Equivalent Transformation
Initial	4	8	48	—
Add 30	7	8	78	Making five: $5 - 2 = 3$
Then add 7	8	5	85	Carry ten and subtract three

Why do these changes preserve the correct result? Because $5 - 2 = 3$, $10 - 3 = 7$, and $10 \cdot 10^i = 10^{i+1}$. In other words, the numerical increment represented by each bead-moving step agrees with the arithmetic operation originally intended. Denote the current abacus configuration by state s and the number it represents by $\mathrm{val}(s)$. If the next step is to add a, a correct bead-position update F must satisfy

$$\mathrm{val}\big(F(s,a)\big) = \mathrm{val}(s) + a. \tag{7}$$

This relationship can be checked step by step in a specific calculation. If every step satisfies it, the result of a sequence of calculations has a basis for its correctness. However, Equation (7) is only a condition for checking whether an operation is correct; it does not mean that all abacus mnemonics have already been individually proved. To summarize this process, I borrow three terms from modern mathematics and computing science that were not expressions used in the ancient book: “place-value encoding” records a number such as 48 through the combination of rods and beads; “equivalent transformation” uses arithmetic identities such as $3 = 5 - 2$ and $7 = 10 - 3$ to replace an addition that is difficult to perform directly with executable bead movements; and “state update” describes the abacus configuration changing from 48 to 78 and then to 85, with each new configuration becoming the starting point for the next step. Thus, first recording a number, then transforming it according to equivalent rules, and then storing the new result can be summarized as “place-value encoding—equivalent transformation—state update.” This analysis does not equate the abacus with an electronic computer; it uses the language of modern science to explain why the abacus can calculate correctly according to definite rules.

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4. The Abacus in the Mind: A Cognitive-Science Explanation of Mental Abacus

An abacus records intermediate results through bead positions, but why can trained individuals continue calculating after the physical abacus has been removed? Following the analysis in the previous section, this question can be divided into two parts: how does the mind retain bead positions, and how does it continue updating them?

Psychological research provides evidence. As early as 1984, Stigler investigated abacus training and mental calculation performance in Chinese children [4]. Frank and Barner subsequently found that skilled mental abacus users can organize numerical values into spatially structured representations; their calculations are relatively resistant to verbal interference, consistent with an explanation based on visuospatial working memory [5]. Visuospatial working memory is the ability to retain and manipulate information about shapes, positions, and similar features over a short period. A multidigit number need not simply be a string of digits to be rehearsed silently; it may also be organized as several adjacent bead positions.

However, "remembering what the abacus looks like" is still insufficient to explain the calculation process. In 2017, Donlan and Wu studied complement operations on the Japanese-style abacus and found that, for trained children, problems requiring more complement steps generally took longer; a control group with no abacus experience did not show the same step effect [6]. This indicates that mental calculation involves not only spatial representations but also retains some trained procedural operations. In the mind, bead positions seem to serve as a way of recording the current state, while complement rules are responsible for changing it into a new state.

This makes the connection between mental abacus and physical abacus calculation concrete: both share decimal place value and complement relationships, while differing in the medium that supports their numerical representations and operating procedures. Physical bead movements and mental activity are, of course, different things; nevertheless, behavioral experiments can at least test whether mental calculators use internal strategies consistent with the structure of bead positions and the steps of complement operations.

5. Traditional Algorithms and Modern Science

Returning to the starting point of this investigation, what I truly wish to explain is this: why can both the traditional algorithms compiled by Cheng Dawei and present-day mental abacus calculation find testable explanations within modern science? The preceding investigation gradually helped me see that such explanations operate at three interconnected yet distinct levels.

The first level concerns mathematical relationships. In the square-root problem, the traditional remainder rule can be explained through linear interpolation, and its error can be calculated directly using Equation (4); excess-and-deficit problems can be explained through the elimination of unknowns in equations. What matters here is not attaching modern terminology to an ancient book, but that the results of traditional calculations can be independently verified through algebraic relationships. Ancient and modern mathematical expressions differ, but this does not alter quantitative facts such as $39^2 - 38^2 = 77$ or $5x - y = 6$.

The second level concerns algorithmic structure. The abacus stores numbers through place-value encoding and makes local updates through making five, complementing to ten, and carrying. Modern computing science likewise studies how data are represented, how algorithms are executed step by step, and how each step guarantees a correct result. Computers usually use binary notation, while the abacus uses decimal bead positions; their materials and operating mechanisms are entirely different, but both can be described in terms of "current

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state—rule-based update—output result." This correspondence rests on explicit mathematical abstraction and does not imply that modern computers developed directly from the abacus.

The third level concerns cognitive mechanisms. Mental abacus users can continue calculating without a physical device; behavioral experiments support the involvement of spatial representations and complement procedures [5-6], and controlled studies also provide evidence for some arithmetic benefits of training [7]. These findings give "practice makes perfect" a more specific meaning in terms of measurable processes of memory, attention, and operation. Modern neuroscience continues to investigate which brain systems participate in this skill and why different students perform differently [8].

Taken together, these three explanations do not amount to the claim that "ancient people had already invented modern science." Instead, they support a more grounded understanding: traditional algorithms have provable mathematical foundations, abacus calculation has an analyzable procedural structure, and mental abacus calculation has testable cognitive mechanisms. This is where modern science and traditional computational knowledge can truly enter into dialogue.

6. Questions Awaiting Explanation and Limitations of the Investigation

At first, I thought that writing an ancient algorithm in modern formulas and proving that its result was correct would amount to understanding it completely. Further investigation showed me that explaining why an algorithm works is not the same as determining how it originated and how people learn to master it.

The square-brick problem that began this investigation is one example. The traditional method uses 77, the difference between consecutive squares, to handle the remainder 17, yielding the approximation $38 + 17/77$. Using modern calculus to approximate the square-root function by its tangent at $x = 1444$ would instead give $38 + 17/76$. The former estimate is too small, while the latter is too large. Modern mathematics can explain the origins of both and compare their errors, but formulas alone cannot determine how ancient practitioners first thought of "doubling the root and adding one." Nor can they reveal the practical use and compilation through which this rule eventually became a mnemonic in a mathematical treatise. Answering these questions requires evidence from editions of ancient texts, historical calculating tools, and actual operating practices. Mathematics can establish the soundness of the traditional method, but cannot by itself reconstruct the entire process of its formation.

Mental abacus raises another question that has not been fully resolved. Modern cognitive experiments show that skilled users' calculations involve visuospatial memory and complement operations, and brain-imaging research has also identified related neural activity. Yet suppose two skilled users can both quickly calculate $48 + 37$: must they necessarily go through the same mental process? One may rely primarily on clear representations of bead positions; might the other rely more heavily on operating procedures made highly fluent by long-term training? How much do visual representation and motor simulation each contribute, and how do they gradually develop with training?

Existing research is attempting to answer these questions. Recent empirical studies have found that certain brain structures and functional connections are associated with training performance or training gains [8-9]. Another controlled study found that mental abacus training could improve some arithmetic performance without a corresponding improvement in general cognitive ability [7]. This suggests to me that the advantages of mental abacus cannot simply be attributed to an overall strengthening of cognitive ability; further research is needed to determine exactly how training changes the representation of numbers and computational processes in the mind. Even identifying relevant brain regions or predicting learning

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performance does not mean that every step of an individual's mental calculation has been reconstructed.

This investigation has therefore made me aware that modern scientific explanations of traditional algorithms have different levels: mathematics can prove the correctness of operating rules, computing science can analyze their execution, and cognitive science can experimentally investigate how people master them. Yet questions still await fuller evidence, from the historical formation of algorithms to the specific working mechanisms of the mental abacus. The absence of a complete explanation does not mean that traditional algorithms lie beyond the scope of scientific investigation; on the contrary, these unanswered questions provide new directions for further research into traditional mathematical knowledge.

7. Conclusion

In 2013, Chinese zhusuan was inscribed on UNESCO's Representative List of the Intangible Cultural Heritage of Humanity [10]. This inquiry-based learning experience has taught me that safeguarding traditional knowledge involves not only preserving abacuses and mnemonics, but also understanding anew why they work and what scientific questions they can still raise today.

What first attracted me was simply the very unusual denominator "77." I first tried to verify the answer, then transformed the specific example into a general formula, plotted the function, and proved the error. Next, I returned to the ancient book to investigate how abacus calculation represents and updates numbers, and drew on experimental research to understand how mental abacus calculation continues some of these operations in the mind. By the end, I found that I had learned not only several ancient algorithms, but also a way of approaching problems: remain curious about unfamiliar answers, use intuition to conjecture conclusions, establish proofs through rigorous calculation and search the literature widely, and then arrive at new questions. This is an upward-spiraling process. Traditional culture and modern science need not be connected through grand narratives of "inventions ahead of their time"; within a single calculation, an operation, and a testable explanation, they can already illuminate one another.

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